Two truncated identities of Gauss

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Abstract. Two new expansions for partial sums of Gauss' triangular and square numbers series are given. As a consequence, we derive a family of inequalities for the overpartition function $\overline{p}(n)$ and for the partition function $\mathrm{pod}(n)$ counting the partitions of n with distinct odd parts. Some further inequalities for variations of partition function are proposed as conjectures.

Keywords: Partition function; Overpartition function; Gauss' identities; Shanks' identity

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1 Introduction

The partition function $p(n)$ has the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \cdots .$$

Two classical results in the partition theory [3, p. 11] are Euler's pentagonal number theorem

$$1 + \sum_{j=1}^{\infty} (-1)^j (q^{j(3j-1)/2} + q^{j(3j+1)/2}) = \prod_{n=1}^{\infty} (1-q^n), \tag{1.1}$$

and Euler's recursive formula for computing $p(n)$:

$$p(n) + \sum_{j=1}^{\infty} (-1)^j (p(n-j(3j-1)/2) + p(n-j(3j+1)/2)) = 0, \tag{1.2}$$

where $p(m) = 0$ for all negative m.

Recently, Merca [11] stumbled upon the following inequality:

$$p(n) - p(n-1) - p(n-2) + p(n-5) \leqslant 0, \tag{1.3}$$

and then, Andrews and Merca [5] proved more generally that, for $k \geqslant 1$,

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j(p(n-j(3j+1)/2)-p(n-(j+1)(3j+2)/2)) \geqslant 0 \tag{1.4}$$

with strict inequality if $n \geqslant k(3k+1)/2$.

The q-shifted factorial and q-binomial coefficient are defined by

$$(a)_\infty = (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad (a)_M = (a;q)_M = \frac{(a;q)_\infty}{(aq^M;q)_\infty},$$

and

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix}_q = \frac{(q;q)_M}{(q;q)_N(q;q)_{M-N}}.$$

Whenever the base of a q-shifted factorial or q-binomial coefficient is just q it will be omitted. The proof of (1.4) in [5] is based on the truncated formula of (1.1):

$$\frac{1}{(q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(3j+1)/2}(1-q^{2j+1}) = 1+(-1)^{k-1}\sum_{n=k}^{\infty}\frac{q^{(k+1)n+\binom{k}{2}}}{(q)_n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Motivated by Andrews and Merca's work [5], in this paper we shall prove new truncated forms of two identities of Gauss [3, p. 23]:

$$1+2\sum_{j=1}^{\infty}(-1)^j q^{j^2} = \frac{(q)_\infty}{(-q)_\infty},$$

$$\sum_{j=0}^{\infty}(-1)^j q^{j(2j+1)}(1-q^{2j+1}) = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty},$$

and derive similar overpartition function and special partition function inequalities.

Theorem 1.1. *For $|q|<1$ and $k \geqslant 1$, there holds*

$$\frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\right) = 1+(-1)^k\sum_{n=k+1}^{\infty}\frac{(-q)_k(-1)_{n-k}q^{(k+1)n}}{(q)_n}\begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{1.5}$$

The overpartition function $\overline{p}(n)$, for $n \geqslant 1$, denotes the number of ways of writing the integer n as a sum of positive integers in non-increasing order in which the first occurrence of an integer may be overlined or not, and $\overline{p}(0)=1$ (see Corteel and Lovejoy [8]). It is easy to see that

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n = \frac{(-q)_\infty}{(q)_\infty} = 1+2q+4q^2+8q^3+14q^4+24q^5+40q^6+\cdots. \tag{1.6}$$

Corollary 1.2. *For $n, k \geqslant 1$, there holds*

$$(-1)^k\left(\overline{p}(n)+2\sum_{j=1}^{k}(-1)^j\overline{p}(n-j^2)\right)\geqslant 0 \tag{1.7}$$

with strict inequality if $n \geqslant (k+1)^2$. For example,

$$\begin{aligned}\overline{p}(n)-2\overline{p}(n-1)&\leqslant 0,\\ \overline{p}(n)-2\overline{p}(n-1)+2\overline{p}(n-4)&\geqslant 0,\\ \overline{p}(n)-2\overline{p}(n-1)+2\overline{p}(n-4)-2\overline{p}(n-9)&\leqslant 0.\end{aligned} \tag{1.8}$$

Theorem 1.3. *For $|q|<1$ and $k \geqslant 1$, there holds*

$$\begin{aligned}&\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(2j+1)}(1-q^{2j+1})\\ &=1+(-1)^{k-1}\sum_{n=k}^{\infty}\frac{(-q;q^2)_k(-q;q^2)_{n-k}q^{2(k+1)n-k}}{(q^2;q^2)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_{q^2}.\end{aligned}$$

Following Hirschhorn and Sellers [10] we denote by $\mathrm{pod}(n)$ the number of partitions of n wherein odd parts are distinct. It is easy to see that

$$\sum_{n=0}^{\infty}\mathrm{pod}(n)q^n=\prod_{n=1}^{\infty}\frac{1+q^{2n-1}}{1-q^{2n}}=1+q+q^2+2q^3+3q^4+4q^5+5q^6+7q^7+\cdots.$$

Corollary 1.4. *For $n, k \geqslant 1$, there holds*

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(\mathrm{pod}\left(n-j(2j+1)\right)-\mathrm{pod}\left(n-(j+1)(2j+1)\right)\Big)\geqslant 0 \tag{1.9}$$

with strict inequality if $n \geqslant (2k+1)k$. For example,

$$\begin{aligned}\mathrm{pod}(n)-\mathrm{pod}(n-1)-\mathrm{pod}(n-3)+\mathrm{pod}(n-6)&\leqslant 0,\\ \mathrm{pod}(n)-\mathrm{pod}(n-1)-\mathrm{pod}(n-3)+\mathrm{pod}(n-6)+\mathrm{pod}(n-10)-\mathrm{pod}(n-15)&\geqslant 0.\end{aligned}$$

A nice combinatorial proof of (1.2) was given by Bressoud and Zeilberger [6]. It would be interesting to find a combinatorial proof of (1.4), (1.7) and (1.9). Moreover, Andrews and Merca [5] found a partition-theoretic interpretation of the truncated sum (1.4). It is still an open problem to give partition interpretations for our two truncated sums in (1.7) and (1.9). A combinatorial proof of (1.8) will be given in Section 4.

2 Proof of Theorem 1.1

Generalizing Shanks' work [12, 13], Andrews [2, Lemma 2] (see also Andrews, Goulden, and Jackson [4, Theorem 1]) established the following identity

$$\sum_{j=0}^{n} \frac{(b)_j(1-bq^{2j})(b/a)_j a^j q^{j^2}}{(1-b)(q)_j(aq)_j} = \frac{(bq)_n}{(aq)_n} \sum_{j=0}^{n} \frac{(b/a)_j a^j q^{(n+1)j}}{(q)_j}. \tag{2.1}$$

When $b=1$ and $a=-1$, the identity (2.1) reduces to

$$1 + 2\sum_{j=1}^{n}(-1)^j q^{j^2} = \sum_{j=0}^{n}(-1)^j \frac{(-1)_j(q)_n q^{(n+1)j}}{(q)_j(-q)_n}. \tag{2.2}$$

By (2.2) and the q-binomial theorem (see [3, Theorem 2.1]), we have

$$\begin{aligned}\frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\right) &= \sum_{j=0}^{k}(-1)^j \frac{(-1)_j(-q^{k+1})_\infty q^{(k+1)j}}{(q)_j(q^{k+1})_\infty} \\ &= \sum_{j=0}^{k}(-1)^j \sum_{i=0}^{\infty} \frac{(-1)_j(-1)_i q^{(k+1)(i+j)}}{(q)_j(q)_i}. \end{aligned} \tag{2.3}$$

After making a change of variable $i+j=n$ and reordering the summation on the right-hand side of (2.3), one should then get a double sum $\sum_{n=0}^{\infty}\sum_{j=0}^{\min\{n,k\}}$. Since $1/(q)_m = 0$ for $m=-1,-2,\dots$, one can write this double sum as

$$\sum_{n=0}^{\infty}\sum_{j=0}^{k}(-1)^j \frac{(-1)_j(-1)_{n-j} q^{(k+1)n}}{(q)_j(q)_{n-j}}.$$

By induction on k, it is easy to see that, for $n \geqslant 1$,

$$\sum_{j=0}^{k}(-1)^j \frac{(-1)_j(-1)_{n-j}}{(q)_j(q)_{n-j}} = (-1)^k \frac{(-q)_k(-1)_{n-k}}{(1-q^n)(q)_{n-k-1}(q)_k}.$$

Hence, the right-hand side of (2.3) can be written as

$$\begin{aligned} &1 + (-1)^k \sum_{n=1}^{\infty} \frac{(-q)_k(-1)_{n-k} q^{(k+1)n}}{(1-q^n)(q)_{n-k-1}(q)_k} \\ &= 1 + (-1)^k \sum_{n=k+1}^{\infty} \frac{(-q)_k(-1)_{n-k} q^{(k+1)n}}{(q)_n} \begin{bmatrix} n-1 \\ k \end{bmatrix}, \end{aligned}$$

as desired.

3 Proof of Corollary 1.2

By (1.5) and (1.6), we see that the generating function for the sequence $\{s_n\}_{n\geqslant 0}$, where

$$s_n = (-1)^k\left(\overline{p}(n) + 2\sum_{j=1}^{k}(-1)^j\overline{p}(n-j^2)\right),$$

is given by

$$(-1)^k + \sum_{n=k+1}^{\infty}\frac{(-q)_k(-1)_{n-k}q^{(k+1)n}}{(q)_n}\begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{3.1}$$

It follows that the coefficient of q^m in (3.1) is nonnegative for $m \geqslant 1$ and positive for $m \geqslant (k+1)^2$. This completes the proof.

4 A combinatorial proof of (1.8)

Let $\mathscr{P}_n$ denote the set of all overpartitions of n. We now construct a mapping $\phi\colon \mathscr{P}_n \to \mathscr{P}_{n-1}$ as follows: For any $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathscr{P}_n$, let

$$\phi(\lambda) = \begin{cases} (\lambda_1, \dots, \lambda_{k-1}), & \text{if } \lambda_k = 1, \\ (\lambda_1, \dots, \lambda_{k-1}, \lambda_k - 1), & \text{if } \lambda_k \neq 1, \overline{1}, \\ (\lambda_1, \dots, \lambda_{k-2}, \underbrace{\hat{1}, \dots, 1}_{\lambda_{k-1}\ 1\text{'s}}), & \text{if } \lambda_k = \overline{1}, \end{cases}$$

where $\hat{1} = \overline{1}$ if λ_{k-1} is overlined and $\hat{1} = 1$ otherwise.

For example, for $n = 4$, the mapping ϕ gives

$$\begin{gathered} 4 \mapsto 3,\ \overline{4} \mapsto \overline{3},\ (3,1) \mapsto 3,\ (3,\overline{1}) \mapsto (1,1,1),\ (\overline{3},\overline{1}) \mapsto (\overline{1},1,1),\ (\overline{3},1) \mapsto \overline{3}, \\ (2,2) \mapsto (2,1),\ (\overline{2},2) \mapsto (\overline{2},1),\ (2,1,1) \mapsto (2,1),\ (2,\overline{1},1) \mapsto (2,\overline{1}),\ (\overline{2},\overline{1},1) \mapsto (\overline{2},\overline{1}), \\ (\overline{2},1,1) \mapsto (\overline{2},1),\ (1,1,1,1) \mapsto (1,1,1),\ (\overline{1},1,1,1) \mapsto (\overline{1},1,1). \end{gathered}$$

It is easy to see that $1 \leqslant |\phi^{-1}(\mu)| \leqslant 2$ for any $\mu \in \mathscr{P}_{n-1}$. This proves that $\overline{p}(n) \leqslant 2\overline{p}(n-1)$.

5 Proof of Theorem 1.3 and Corollary 1.4

In [12, 13] Shanks proved that

$$\sum_{j=0}^{n-1} q^{j(2j+1)}(1+q^{2j+1}) = \sum_{j=0}^{n-1}\frac{(q;q^2)_j(q^2;q^2)_n q^{j(2n+1)}}{(q^2;q^2)_j(q;q^2)_n}. \tag{5.1}$$

By (5.1) (with q replaced by $-q$) and the q-binomial theorem (see [3, Theorem 2.1]), we have

$$
\begin{aligned}
&\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(2j+1)}(1-q^{2j+1})\\
&\quad=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=0}^{k-1}(-1)^j\frac{(-q;q^2)_j(q^2;q^2)_k q^{(2k+1)j}}{(q^2;q^2)_j(-q;q^2)_k}\\
&\quad=\sum_{j=0}^{k-1}(-1)^j\frac{(-q;q^2)_j(-q^{2k+1};q^2)_\infty q^{(2k+1)j}}{(q^2;q^2)_j(q^{2k+2};q^2)_\infty}\\
&\quad=\sum_{j=0}^{k-1}(-1)^j\sum_{i=0}^{\infty}\frac{(-q;q^2)_j(-q^{-1};q^2)_i q^{(2k+1)(i+j)+i}}{(q^2;q^2)_j(q^2;q^2)_i}.
\end{aligned}
\tag{5.2}
$$

By induction on k, it is easy to see that, for $n\geqslant 1$,

$$
\sum_{j=0}^{k-1}(-1)^j\frac{(-q;q^2)_j(-q^{-1};q^2)_{n-j}q^{n-j}}{(q^2;q^2)_j(q^2;q^2)_{n-j}}=(-1)^{k-1}\frac{(-q;q^2)_k(-q;q^2)_{n-k}q^{n-k}}{(1-q^{2n})(q^2;q^2)_{n-k}(q^2;q^2)_{k-1}}.
$$

Hence, letting $i+j=n$, the right-hand side of (5.2) can be written as

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{j=0}^{k-1}(-1)^j\frac{(-q;q^2)_j(-q^{-1};q^2)_{n-j}q^{(2k+1)n+n-j}}{(q^2;q^2)_j(q^2;q^2)_{n-j}}\\
&\quad=1+(-1)^{k-1}\sum_{n=1}^{\infty}\frac{(-q;q^2)_k(-q;q^2)_{n-k}q^{2(k+1)n-k}}{(1-q^{2n})(q^2;q^2)_{n-k}(q^2;q^2)_{k-1}}.\\
&\quad=1+(-1)^{k-1}\sum_{n=k}^{\infty}\frac{(-q;q^2)_k(-q;q^2)_{n-k}q^{2(k+1)n-k}}{(q^2;q^2)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_{q^2}.
\end{aligned}
$$

This proves Theorem 1.3. The proof of Corollary 1.4 is similar to that of Corollary 1.2 and is omitted here.

6 Open problems

In this section, we propose a common generalization of (1.4), (1.7) and (1.9). Let m,r be positive integers with $1\leqslant r\leqslant m/2$. Consider the generalized partition function $J_{m,r}(n)$ defined by

$$
\sum_{n=0}^{\infty}J_{m,r}(n)q^n=\frac{1}{(q^r,q^{m-r},q^m;q^m)_\infty},
\tag{6.1}
$$

where

$$
(a,b,c;q)_\infty=(a;q)_\infty(b;q)_\infty(c;q)_\infty.
$$

It is easy to see that

$$J_{2,1}(n)=\overline{p}(n),\quad J_{3,1}(n)=p(n),\quad J_{4,1}(n)=\mathrm{pod}(n).$$

Moreover, if $r<m/2$, then $J_{m,r}(n)$ can be understood as the number of partitions of n into parts congruent to $0,\pm r$ modulo m. Now, Jacobi's triple product identity implies (see [9, p. 375]) that

$$1+\sum_{j=1}^{\infty}(-1)^j(q^{j(mj+m-2r)/2}+q^{j(mj-m+2r)/2})=(q^r,q^{m-r},q^m;q^m)_\infty. \tag{6.2}$$

It follows from (6.1) and (6.2) that $J_{m,r}(n)$ satisfies the recurrence formula:

$$J_{m,r}(n)+\sum_{j=1}^{\infty}(-1)^j\Big(J_{m,r}\big(n-j(mj-m+2r)/2\big)+J_{m,r}\big(n-j(mj+m-2r)/2\big)\Big)=0,$$

where $J_{m,r}(s)=0$ for all negative s.

Conjecture 6.1. *For $m,n,k,r\geqslant 1$ with $r\leqslant m/2$, there holds*

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(J_{m,r}\big(n-j(mj+m-2r)/2\big)-J_{m,r}\big(n-(j+1)(mj+2r)/2\big)\Big)\geqslant 0 \tag{6.3}$$

with strict inequality if $n\geqslant k(mk+m-2r)/2$.

Remark 6.2. After we posted a preliminary version of this paper on arXiv, George E. Andrews informed us that our Conjecture 6.1 is effectively identical with Problem 2 in the final version of [5].

For $m=2$ and $r=1$, the inequality (6.3) is equivalent to

$$(-1)^{k-1}\left(\overline{p}(n)+2\sum_{j=1}^{k-1}(-1)^j\overline{p}(n-j^2)\right)-\overline{p}(n-k^2)\geqslant 0 \tag{6.4}$$

with strict inequality if $n\geqslant k^2$. It is clear that (6.4) is stronger than the proved inequality (1.7) (with k replaced by $k-1$). By (1.5) and (1.6), the generating function of the left-hand side of (6.4) is equal to

$$\begin{aligned}(-1)^{k-1}&\frac{(-q)_\infty}{(q)_\infty}\left(1+(-1)^kq^{k^2}+2\sum_{j=1}^{k-1}(-1)^jq^{j^2}\right)\\ &=(-1)^{k-1}+\sum_{n=k}^{\infty}\frac{(-q)_{k-1}(-q)_{n-k}q^{kn}}{(q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}\\ &\qquad-\sum_{n=k+1}^{\infty}\frac{(-q)_k(-q)_{n-k-1}q^{(k+1)n}}{(q)_n}\begin{bmatrix}n-1\\k\end{bmatrix}.\end{aligned} \tag{6.5}$$

Therefore, the conjectured inequality (6.4) is equivalent to

Conjecture 6.3. *Let $k \geqslant 1$. Then the coefficient of q^m in the right-hand side of* (6.5) *is nonnegative for $m \geqslant 1$ and positive for $m \geqslant k^2$.*

For $m = 5$ and $r = 1, 2$, the inequality (6.3) may be written as

$$(-1)^{k-1} \sum_{j=-k}^{k-1} (-1)^j J_{5,1}\big(n - j(5j+3)/2\big) \geqslant 0, \tag{6.6}$$

$$(-1)^{k-1} \sum_{j=-k}^{k-1} (-1)^j J_{5,2}\big(n - j(5j+1)/2\big) \geqslant 0. \tag{6.7}$$

By (6.1), the conjectured inequalities (6.6) and (6.7) are equivalent to

$$(-1)^k + \frac{(-1)^{k-1}}{(q, q^4, q^5; q^5)_\infty} \sum_{j=-k}^{k-1} (-1)^j q^{j(5j+3)/2} \in \mathbb{N}[[q]], \tag{6.8}$$

$$(-1)^k + \frac{(-1)^{k-1}}{(q^2, q^3, q^5; q^5)_\infty} \sum_{j=-k}^{k-1} (-1)^j q^{j(5j+1)/2} \in \mathbb{N}[[q]]. \tag{6.9}$$

The two series $\sum_{j=-k}^{k-1}$ in (6.8) and (6.9) already appeared in the works of Andrews [1] and Warnaar [15] (see also Chapman [7]) as partial-sum analogues of Rogers-Ramanujan identities. In particular, they obtained alternative expressions of these series as double sums. However, we have no idea how to use their formulas to tackle the conjectures (6.8) and (6.9).

Along the same line of thinking, we consider the sequence $\{t(n)\}_{n \geqslant 0}$ (see A000716 in Sloane's database of integer sequences [14]) defined by

$$\begin{aligned} \sum_{n=0}^{\infty} t(n) q^n &= \frac{1}{(q)_\infty^3} \\ &= 1 + 3q + 9q^2 + 22q^3 + 51q^4 + 108q^5 + 221q^6 + 429q^7 + 810q^8 + 1479q^9 + \cdots . \end{aligned}$$

Clearly, the number $t(n)$ counts partitions of n into 3 kinds of parts. Now, invoking the identity of Jacobi [9, p. 377]:

$$(q)_\infty^3 = \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2},$$

we derive the recurrence formula:

$$\sum_{j=0}^{\infty} (-1)^j (2j+1) t(n - j(j+1)/2) = 0,$$

where $t(m) = 0$ for all negative m.

We end the paper with the following conjecture:

Conjecture 6.4. *For $n, k \geqslant 1$, there holds*

$$(-1)^k \sum_{j=0}^{k} (-1)^j (2j+1) t(n - j(j+1)/2) \geqslant 0$$

with strict inequality if $n \geqslant (k+1)(k+2)/2$. For example,

$$t(n) - 3t(n-1) \leqslant 0,$$
$$t(n) - 3t(n-1) + 5t(n-3) \geqslant 0,$$
$$t(n) - 3t(n-1) + 5t(n-3) - 7t(n-6) \leqslant 0.$$

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